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Some stability condition of atomic types

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Abstract

In recent years, the results about atomic abstract elementary class were summarized and developed by J.T.Baldin. In his works, categoricity of atomic AEC is one of the main problems under the assumption of atomic $\omega-$stability. I tried the argument around the problem under some weaker conditions.

1. Stability condition and splitting

We recall some definitions at first. In this note, I define the atomic AEC briefly for convenience' sake. Please refer to Baldwin's book [1] for the accurate definition.

Definition 1 Let T be a complete theory of a language L.
A class of structures $(\mathbf{K}, \prec_{\mathbf{K}})$ is an *atomic abstract elementary class* (AEC) if $\mathbf{K}$ is the class of atomic models of T and $\prec_{\mathbf{K}}$ is the first order elementary submodel.
We assume that $\mathbf{K}$ has at least one uncountable atomic model. And the language L is countable.

Definition 2 Let T be a complete first order theory.
A set A contained in a model M of T is *atomic* if every finite sequence in A realizes a principal type over the empty set.
If an atomic set A is a model of T, then we say A is an atomic model.
Let A be an atomic set.
$S_{at}(A)$ is the collection of $p \in S(A)$ such that if $a \in \mathcal{M}$ realizes p, then Aa is atomic where $\mathcal{M}$ is the big model.
We refer to a $p \in S_{at}(A)$ as an *atomic type*.

We consider stability conditions for atomic types.

Definition 3 An atomic AEC $\mathbf{K}$ is $\lambda-$ *stable* if for every $M \in \mathbf{K}$ of cardinality λ, $|S_{at}(M)| = \lambda$.

In Baldwin's book [1], many examples of atomic AEC are shown which satisfy some stability conditions.

The notion of independence by splitting is available in this context.

Definition 4 *Let T be a complete theory of a language L. And let Δ_i be a set of formulas for $i = 1, 2$.*

A (complete) type $p(x)$ over B $(\Delta_1, \Delta_2)-$splits over $A \subset B$ if there are $b, c \in B$ which realize the same Δ_1-type over A and a Δ_2-formula $\phi(x, y)$ such that $\phi(x, b) \in p$ and $\neg\phi(x, c) \in p$.

A (complete) type $p(x)$ over B splits over $A \subset B$ if $p(x)$ $(L, L)-$splits over A.

We confirm the next lemmas. Their proof are some modifications from Shelah's book [12].

Lemma 5 *If $\mathbf{K}$ is atomic $\omega-$stable, then for any $M \in \mathbf{K}$ and any nonalgebraic $p(x) \in S_{at}(M)$, there is no increasing sequence $\{A_i\}_{i<\omega} \subset M$ such that $p \restriction A_{i+1}$ splits over A_i for all $i < \omega$.*

Thus $p(x)$ does not split over some finite $A \subset M$.

Lemma 6 *Let M be a countable atomic model and $\varphi(x, y)$ be a formula.*

Suppose that $|\, S^{\varphi}_{at}(M) \,| > \aleph_0$ and let $\{c_i : i < \aleph_1\}$ be a set of realizations of types in $S^{\varphi}_{at}(M)$.

Then there are $i < \aleph_1$ and $\{A_j : j < \aleph_0\} \subset M$ such that

for any $j < \aleph_0$, $|\, A_j \,| < \aleph_0$ and

$\mathrm{tp}^{\varphi}_{at}(c_i/A_{j+1})$ $(\psi, \varphi)-$splits over A_j for any $j < \aleph_0$ where $\psi(y, x) = \varphi(x, y)$.

2. Atomic AEC without infinite splitting chain

Before I tried to argue about categoricity problem of atomic AEC under the asssumption that any atomic model $M \in \mathbf{K}$ has no infinite splitting chain for complete types. But shortly afterward, it came out that the condition of no infinite splitting chain is not weaker than atomic $\omega-$stability essentially. Thus we consider some weaker conditions.

Definition 7 Let $\mathbf{K}$ be an atomic AEC and $M \in \mathbf{K}$.

And let $\Delta(x, y)$ be the set of complete formulas such that $\Delta(x, y) = \{\varphi_i(x, y) : \varphi_i$ is complete and $\mathrm{tp}_{at}(y/\emptyset)$ is unique for $i < \omega\}$.

M has no infinite splitting chain if for any formula $\varphi(x, y)$ and $p \in S^{\varphi}_{at}(M)$, there is no increasing sequence $\{A_i\}_{i<\omega} \subset M$ such that $p \restriction A_{i+1}$ splits over A_i for all $i < \omega$, and moreover,

for any set of complete formulas $\Delta(x, z)$ as above and for any $q \in S^{\Delta}_{at}(M)$, there is no increasing sequence $\{A_i\}_{i<\omega} \subset M$ such that $q \upharpoonright A_{i+1}$ splits over A_i for all $i < \omega$.

K *has no infinite splitting chain* if any model $M \in \mathbf{K}$ has no infinite splitting chain.

We can prove the next lemma.

Lemma 8 *Let M have no infinite splitting chain.*

Then for any $p(x) \in S_{at}(M)$ for some $M \in \mathbf{K}$, if $M \subset B$ and B is atomic, then there is a unique extension of p to $\hat{p} \in S_{at}(B)$ which does not split over some countable set $C \subset M$.

Remark 9 *There is an example M satisfying ; for any formula $\varphi(x, y)$ and $p \in S^{\varphi}_{at}(M)$, p has no infinite splitting chain, and for some atomic $B \supset M$ and $q \in S^{\psi}_{at}(M)$, q has no atomic extension over B.*

Lemma 10 *Under the assumption that $\mathcal{K}$ has no infinite splitting chain, almost all forking axioms hold for splitting.*

In particular, (restricted) transitivity and symmetry over models.

In this context, we consider Morley sequences constructed by nonsplitting extensions. Thus Morley sequences are indiscernible.

Definition 11 Let $A \subset M \in \mathbf{K}$ and $p(x) \in S_{at}(M)$ be nonalgebraic.

A sequence $I = \{a_i : i < \lambda\}$ in M is a *Morley sequence* of $p(x)$ over A if $p(x)$ does not split over A, and for any $i < \lambda$, $a_i \models p \upharpoonright A$ and $\mathrm{tp}_{at}(a_i/\{a_j : j < i\} \cup A)$ does not split over A.

We characterize Morley sequences in this context.

Lemma 12 *If there is $N \in \mathbf{K}$ with $|N| > \aleph_0$ such that N has no infinite splitting chain.*

Then there are $M \in \mathbf{K}$ with $|M| = \aleph_2$ and a nonalgebraic type $q(x) \in S_{at}(M)$ such that M has no infinite splitting chain and q has a Morley sequence I in M with $|I| = \aleph_2$.

Lemma 13 *Let $N \in \mathbf{K}$ with $|N| > \aleph_0$ such that N has no infinite splitting chain.*

Suppose that there is a nonalgebraic type $p(x) \in S_{at}(N)$ such that p has a Morley sequence I in N with $|I| > \aleph_0$.

Then I is totally indiscernible.

By Morley sequences, we can argue about the definability of atomic types.

Lemma 14 *Let $M \in \mathbf{K}$ with $|M| > \aleph_0$ and M has no infinite splitting chain. And let $p(x) \in S_{at}(M)$ be nonalgebraic.*

Suppose that $p(x)$ has a Morley sequence $I = \{a_i : i < \aleph_0\}$ in M.

Then for any $\phi(x, y)$, there is $n_\phi < \aleph_0$ such that for any $b \in M$,

$|\{a_i : M \models \phi(a_i, b),\ i < \aleph_0\}| < n_\phi$

or $|\{a_i : M \models \neg\phi(a_i, b),\ i < \aleph_0\}| < n_\phi$.

Lemma 15 *Let $M \in \mathbf{K}$ and M has no infinite splitting chain. And let $p(x) \in S_{at}(M)$ be nonalgebraic and p have a Morley sequence $I = \{a_i : i < \aleph_0\}$ in M over some countable set $D \subset M$.*

Then $p(x)$ is definable over $D \cup I$, that is,

for any formula $\phi(x, y) \in L$, there is $n_\phi < \omega$ such that for any $b \in M$,

$\phi(x, b) \in p$

if and only if

$\models \bigvee_{w \subset \{0,1,\cdots,2n_\phi\},\, |w| = n_\phi+1} \bigwedge_{i \in w} \phi(a_i, b)$.

Lemma 16 *Let $M \in \mathbf{K}$ with $|M| > \aleph_0$.*

Suppose that there is a nonalgebraic $p(x) \in S_{at}(M)$ such that ;

$p(x)$ does not split over D for some countable set $D \subset M$, and

$p(x)$ has a Morley sequence I of $p(x)$ over D in M with $|I| \geq \aleph_0$.

Then $p(x) \upharpoonright D$ is stationary with respect to nonsplitting extension.

By the previous lemmas, we can characterize atomic ω−stability.

Lemma 17 *Let $\mathbf{K}$ be an atomic AEC.*

Then $\mathbf{K}$ is atomic ω−stable

if and only if

for any $M \in \mathbf{K}$ and any nonalgebraic $p(x) \in S_{at}(M)$, there is no increasing sequence $\{A_i\}_{i<\omega} \subset M$ such that $p \upharpoonright A_{i+1}$ splits over A_i for all $i < \omega$.

3. Existence of pregeometry

In [1], categoricity of atomic AEC is proved by means of the fact that every model is prime and minimal over a basis of some pregeometry given by a quasi-minimal set. Thus we try to define pregeometry in the present context.

We recall the definition of pregeometry.

Definition 18 Let X be an infinite set and cl a function from $\mathcal{P}(X)$ to $\mathcal{P}(X)$ where $\mathcal{P}(X)$ denotes the set of all subsets of X. If the function cl satisfies the following properties, we say (X, cl) is *pregeometry*.

(I) $A \subset B \Longrightarrow A \subset \mathrm{cl}(A) \subset \mathrm{cl}(B)$,

(II) $\mathrm{cl}(\mathrm{cl}(A)) = \mathrm{cl}(A)$,

(III) (Finite character) $b \in \mathrm{cl}(A) \implies b \in \mathrm{cl}(A_0)$ for some finite $A_0 \subset A$,
(IV) (Exchange axiom)
$b \in \mathrm{cl}(A \cup \{c\}) - \mathrm{cl}(A) \implies c \in \mathrm{cl}(A \cup \{b\})$.

We define a big type which is a (modified) notion in [1].

Definition 19 Let $a \in M$ and $A \subset M \in \mathbf{K}$.
A nonalgebraic atomic type $\mathrm{tp}_{at}(a/A)$ is *big* if there is an atomic model $N \in \mathbf{K}$ such that $A \subset N$ and $\mathrm{tp}_{at}(a/A)$ has a nonalgebraic atomic extension over N.

Next we show the existence of types with rank one, i.e. quasi-minimal types.

Lemma 20 *Let* $\mathbf{K}$ *have no infinite splitting chain and* $M \in \mathbf{K}$ *with* $|M| > \aleph_0$. *And let* $p(x) \in S_{at}(M)$ *be nonalgebraic and* $p(x)$ *does not split over* B *for some countable set* $B \subset M$.
Then there are an atomic model N *with* $|N| = \aleph_2$ *and an extension* $q(x) \in S_{at}(N)$ *of* $p \restriction B$ *such that for some countable set* C *with* $B \subset C \subset M$, $q \restriction C$ *is big, but any splitting extension of* $q \restriction C$ *over* N *is not big, and* $q(x)$ *has a Morley sequence* I *over* C *in* N *with* $|I| = \aleph_2$.

We define some closure operator.

Definition 21 Let $M \in \mathbf{K}$ and $p(x) \in S^1_{at}(M)$. And let p does not split over some countable parameter $A \subset M$ and $p \restriction A$ is stationary.
The operator cl_p is defined for any $X \subset (p \restriction A)(M)$ by ;
$cl^0_p(X) = X$ and $cl^{n+1}_p(X) = \{ a \in (p \restriction A)(M) \mid a \notin (p \restriction cl^n_p(X) \cup A)(M)\}$, and $cl_p(X) = \bigcup_{n<\omega} cl^n_p(X)$.

We can prove the next fact.

Theorem 22 *Let* $M \in \mathbf{K}$ *with* $|M| > \aleph_0$ *and* M *has no infinite splitting chain.*
Then there are an uncountable model N *and* $p(x) \in S^1_{at}(N)$ *such that* $p(x)$ *does not split over some countable set* $D \subset N$ *and* $((p \restriction D)(N), cl_p)$ *is pregeometry.*

4. Constructible sequence of atomic types

On the proof of categoricity of $*$-excellent atomic AEC $\mathbf{K}$, the existence of prime model plays a crucial role. In this section, we show the existence of some constructible model under the assumption that $\mathbf{K}$ has no infinite splitting chain and has some additional conditions. After that we replace prime models by these constructible models in the argument of local categoricity.

At first we define some isolation of atomic types.

Definition 23 Let $a \in M \in \mathbf{K}$ and $A \subset M$.

A type $\mathrm{tp}_{at}(a/A)$ is *finitely isolated* if there is $\bar{b} \in M$ such that $\mathrm{tp}_{at}(a/\bar{b}) \vdash \mathrm{tp}_{at}(a/A)$.

A sequence $\{c_i : i < \alpha\} \subset M$ is *finitely constructible over* A if, for any $\beta < \alpha$, $\mathrm{tp}_{at}(c_\beta/A \cup \{c_i : i < \beta\})$ is finitely isolated.

M is *finitely constructible over* A if $M \setminus A$ can be written as a finitely constructible sequence.

An atomic set C with $A \subset C \subset M$ is *quasi – atomic over* A if for any $\bar{c} \in C$, $\mathrm{tp}_{at}(\bar{c}/A)$ is finitely isolated.

We confirm the next lemma.

Lemma 24 *Let $N \in \mathbf{K}$ and $A \subset C \subset N$.*

If $C_\beta = \{c_\alpha : \alpha < \beta\}$ is finitely constructible over A, then $C_\beta \cup A$ is quasi-atomic over A.

So far, we argue under the assumption that $\mathbf{K}$ has no infinite splitting chain. But additional condition is necessary to show the existence of finitely constructible models.

Definition 25 Let $\mathbf{K}$ be an atomic AEC and $M \in \mathbf{K}$.

Let $\Delta(x, y)$ be a set of complete formulas such that $\Delta(x, y) = \{\varphi_i(x, y) : \varphi_i$ is complete and $\mathrm{tp}_{at}(y/\emptyset)$ is unique for $i \in I\}$.

Let $A \subset B \subset M$ such that B is quasi-atomic over A and A is totally indiscernible (over $\emptyset$), for these sets, let $\mathrm{tp}_{at}(B, A) = \Phi(Y, Z)$, and

let $\Gamma(x, Y, Z)$ be a set of complete formulas for a quasi-atomic parameter over a totally indiscernible sequence such that $\Gamma(x, Y, Z) = \{\psi_i(x, \bar{y}_i, \bar{z}_i) : \psi_i(x, \bar{y}_i, \bar{z}_i)$ is complete and $\bar{y}_i$ is some enumeration of $\{y_{i_j} : j < k\} \subset Y = (y_i)_{i<\lambda}$, and $\bar{z}_i$ is some enumeration of $\{z_{i_j} : j < l\} \subset Z = (z_i)_{i<\mu}\}$.

M has no infinite splitting chain for quasi – atomic parameters (over totally indiscernible sequences) if for any set of complete formulas $\Delta(x, y)$ and for any nonalgebraic $p \in S^{\Delta}_{at}(M)$, there is no increasing sequence $\{A_i\}_{i<\omega} \subset M$ such that $p \restriction A_{i+1}$ splits over A_i for all $i < \omega$, and,

for any set of complete formulas $\Gamma(x, Y, Z)$ for quasi-atomic parameters over totally indiscernible sequences and for any nonalgebraic $q \in S^{\Gamma}_{at}(M)$, there is no increasing sequence $\{A_i\}_{i<\omega} \subset M$ such that $q \restriction A_{i+1}$ splits over A_i for all $i < \omega$.

$\mathbf{K}$ *has no infinite splitting chain for quasi – atomic parameters* if any model $M \in \mathbf{K}$ has no infinite splitting chain for quasi-atomic parameters.

We show the next lemma.

Lemma 26 *($\mathbf{K}$ has no infinite splitting chain.)*

Let $M \in \mathbf{K}$ and $\Delta(x) = \{\phi_i(x, y_i) : i \in I\}$ for some set I. And let $A \subset B \subset M$ and a be such that $\mathrm{tp}^{\Delta}_{at}(a/B)$ does not split over A and $\mathrm{tp}^{\Delta}_{at}(a/A)$ is *stationary.*

Then the following are equivalent ;

(i) $\mathrm{tp}_{at}^{\Delta}(a/A) \vdash \mathrm{tp}_{at}^{\Delta}(a/B)$

(ii) For any a' *with* $\mathrm{tp}_{at}^{\Delta}(a'/A) = \mathrm{tp}_{at}^{\Delta}(a/A)$, *if* $\mathrm{tp}^{\Delta}(a'/B)$ *is atomic, then* $\mathrm{tp}_{at}^{\Delta}(a'/B)$ *does not split over* A.

We prove the existence of some constructible model over totally indiscernible sequences under the assumption that $\mathbf{K}$ has no infinite splitting chain for quasi-atomic parameters. But we does not assume that the type of element in indiscernible sequences has rank one.

Proposition 27 *Let* $N \in \mathbf{K}$ *with* $|N| > \aleph_0$ *and* N *has no infinite splitting chain for quasi-atomic parameters. And let* $I \subset A \subset N$, *and* $I = \{c_i : i < \lambda\}$ *be a totally indiscernible sequence (over* $\emptyset$ *) and* A *be a quasi-atomic set over* I.

Suppose that $\phi(x, b)$ *be a* $L(A)-$*formula.*

Then there is $p(x) \in S_{at}(A)$ *such that* $\phi(x, b) \in p$ *and* p *is finitely isolated.*

Proposition 28 *Let* $N \in \mathbf{K}$ *with* $|N| > \aleph_0$ *and* N *has no infinite splitting chain for quasi-atomic parameters. And let* $I \subset N$ *be a totally indiscernible sequence (over* $\emptyset$ *).*

Then there is $M \prec N \in \mathbf{K}$ *such that* $I \subset M$ *and* M *is finitely constructible over* I.

Theorem 29 *Let* $N \in \mathbf{K}$ *with* $|N| > \aleph_0$ *and* N *has no infinite splitting chain for quasi-atomic parameters. And let* $I \subset N$ *be a totally indiscernible sequence and* $M \prec N \in \mathbf{K}$ *be finitely constructible over* I.

Then for any $M' \prec N$ *with* $I \subset M'$, M *is elementarily embedded in* M' *over* I.

Even if there are two finitely constructible models over some set $A \subset M \in \mathbf{K}$, we can not prove the isomorphism of them instantly. Another condition is need to prove the isomorphism. On the proof of categoricity of $^*-$excellent atomic AEC $\mathbf{K}$, they set the condition that $\mathbf{K}$ has no Vaughtian triple. Thus we must set this condition to prove the isomorphism of finitely constructible models over a basis of pregeometry.

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